2020/12/02
G3

TD6 : Ex5. Ex6. Ex7) 60'

DS : TD3. TD4. TD5.
7/12

TD6 : Ex2. Ex3 : ① $f(x) = \sqrt{1+x+x^2}$. $x_0 = 0$

$f(0)$, $f'(x)$ $f''(x)$ Taylor-Young.

Rappel : DL usuels (pour $x \to 0$)

1. $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + x^n \varepsilon(x)$ ($\varepsilon(x) \to 0$)
2. $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + x^n \varepsilon(x)$
3. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots (-1)^{n-1} x^n + x^n \varepsilon(x)$
4. $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n + x^n \varepsilon(x)$

$f(x) = \sqrt{1+x+x^2}$

posons $u(x) = x + x^2$. $f = \sqrt{1+x+x^2} = \sqrt{1+u} = (1+u)^{\frac{1}{2}}$

$(1+u)^{1/2} = 1 + \frac{1}{2} u + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2} u^2 + u^2 \varepsilon(u)$

$= 1 + \frac{1}{2}(x+x^2) + \left(-\frac{1}{8}\right)\left((x+x^2)^2\right) + x^2 \varepsilon(x)$

$= 1 + \frac{1}{2}x + \frac{1}{2}x^2 - \frac{1}{8}(x^2) + x^2 \varepsilon(x)$

$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + x^2 \varepsilon(x)$

$\boxed{u = x + x^2 \quad u^2 = (x+x^2)^2 = x^2 + \cdots}$

$\left(\frac{1}{2} - \frac{1}{8}\right) = \frac{4}{8} - \frac{1}{8} = \frac{3}{8}$

$x^2 + \underbrace{2x^3 + x^4}_{\varepsilon(x)}$

$1 + ① x + ② x^2 + x^2 \varepsilon(x)$

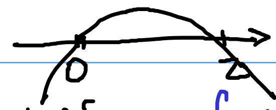
Ex5: ① $f(x) = 3x^2 e^{-x}$ $\quad D_f = \mathbb{R}$ $\quad (uv)' = u'v + uv'$

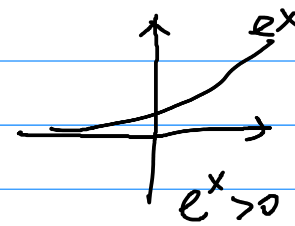
$= 3 \times \frac{x^2}{e^x}$ $\quad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$$f'(x) = 3 \times \frac{2x \times e^x - x^2 . e^x}{(e^x)^2} = 3 \times \frac{2x - x^2}{e^x} = 3 \times \frac{x(2-x)}{e^x}$$

$f'(x)$ a le signe du poly $x \times (2-x)$

$f'(x) = 0$ pour $x = 0$ et $x = 2$





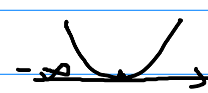
$f'(x) < 0$ pour $x \in]-\infty, 0] \cup [2, +\infty[$, $f(x) \searrow$

$f'(x) > 0$ pour $x \in [0, 2)$. $f(x) \nearrow$

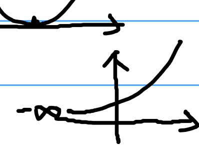
le tableau de variation.

	$-\infty$		0		2		$+\infty$
$f'(x)$		$-$		$+$		$-$	
$f(x)$	$+\infty$	$\searrow$	0	$\nearrow$	$12e^{-2}$	$\searrow$	0

$x^2 \nearrow$



$e^x \nearrow$



$f(0) = 0$

$f(2) = 3 \times \frac{2^2}{e^2} = 12e^{-2}$

$f(x) = 3 \times \frac{x^2}{e^x}$

$\lim_{x \to -\infty} f(x) = +\infty$

$\lim_{x \to +\infty} f(x) = 0$

$\lim_{x \to -\infty} e^x = 0$

Tracer la graphe:

$$f''(x) = 3 \times \frac{(2-2x)e^x - (2x - x^2) . e^x}{(e^x)^2} = 3 \times \frac{2 - 2x - (2x - x^2)}{e^x}$$

$$= 3 \times \frac{2 - 4x + x^2}{e^x}$$

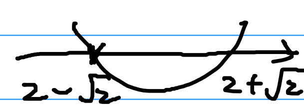
$f''(x)$ a le signe du poly $\underline{2 - 4x + x^2}$

$\Delta = (-4)^2 - 4 \times 1 \times 2 = 8$. $\quad x_1 = \frac{4 + \sqrt{8}}{2} = \frac{4 + 2\sqrt{2}}{2} = 2 + \sqrt{2}$

$x_2 = 2 - \sqrt{2}$

$f''(x) > 0$ pour $x \in]-\infty, 2-\sqrt{2}] \cup [2+\sqrt{2}, +\infty[$, convexe

$f''(x) < 0$ pour $x \in [2-\sqrt{2}, 2+\sqrt{2})$, $f(x)$ concave

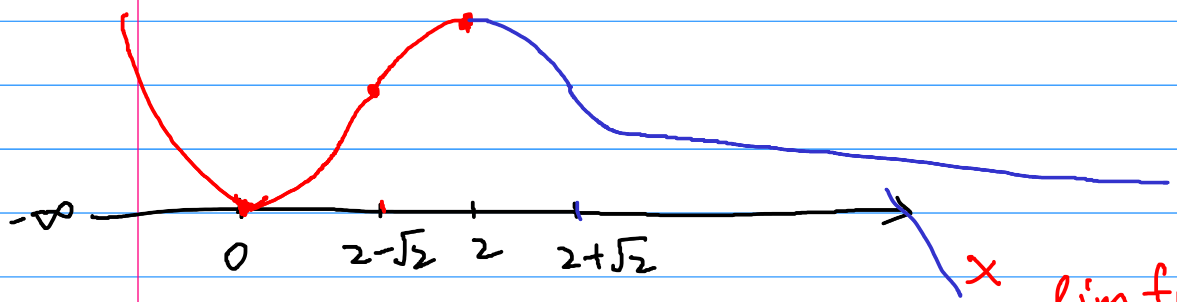


$f''(x) > 0$ pour $x \in]-\infty, 2-\sqrt{2}] \cup [2+\sqrt{2}, +\infty[$, convexe

$f''(x) < 0$ pour $x \in [2-\sqrt{2}, 2+\sqrt{2})$, $f(x)$ concave

$f'(x) < 0$ pour $x \in]-\infty, 0] \cup [2, +\infty[$, $f(x) \searrow$

$f'(x) > 0$ pour $x \in [0, 2)$. $f(x) \nearrow$

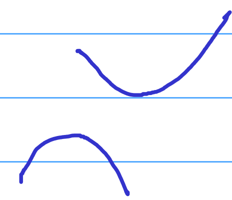


$\lim\limits_{x \to +\infty} f(x) = 0$

$f(0) = 0$ $\qquad f(2) = 12e^{-2}$

$f(x)$: $f''(x) > 0 \Leftrightarrow$ convexe

$f''(x) < 0 \Leftrightarrow$ concave



On conclure que il y a un maximum local en $x = 2$, $f(2) = 12e^{-2}$

pas maximum global.

il y a un minimum global en $x = 0$, $f(0) = 0$.

Ex5 : 3

$$h(x) = 2\ln(\underbrace{1+x^2}_{>0}) \qquad D_f = \mathbb{R}.$$

$$(\ln u)' = \frac{1}{u}$$

$$(\ln u(x))' = \frac{1}{u(x)} \cdot u'(x)$$

$$h'(x) = 2 \times \frac{1}{1+x^2} \times (2x).$$

$$(1+x^2)' = 2x$$

$$= \frac{4x}{1+x^2}.$$

dont le signe est celui de x.

$h'(x) < 0$ pour $x < 0$. donc $h(x) \searrow$

$h'(x) > 0$ pour $x > 0$ donc $h(x) \nearrow$

x	$-\infty$		0		$+\infty$
$h'(x)$		$-$	0	$+$	
$h(x)$	$+\infty$	$\searrow$	0	$\nearrow$	$-\infty$

$h(0) = 2\ln(1+0^2) = 0$

$\lim\limits_{x \to +\infty} h(x) = +\infty$

$\lim\limits_{x \to -\infty} h(x) = +\infty$

On conclure que. $h(x)$ admet un unique minimum global en $x=0$. $h(0)=0$.

Il n'y a pas de maximum global ni local.

Exb:

① . $f(x) = \dfrac{x^2+7}{(x-3)^2}$. $D_f =]-\infty, 3[\cup]3, +\infty[= \mathbb{R} \setminus \{3\}$

$$f'(x) = \frac{2x(x-3)^2 - (x^2+7)(2(x-3))}{(x-3)^{2\times 2}}$$

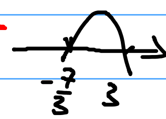
$$= \frac{2x(x-3) - (x^2+7)\times 2}{(x-3)^3} = \frac{\cancel{2x^2} - 6x - \cancel{2x^2} - 14}{(x-3)^3}$$

$$= -2\times \frac{3x+7}{(x-3)^3}$$

$$= 2\times \frac{3x+7}{(3-x)(x-3)^2}$$

$-(x-3) = 3-x$

qui a le signe du poly $(3x+7)(3-x)$



$f'(x) < 0$ pour. $x \in]-\infty, -\frac{7}{3}[\cup [3, +\infty[$

$f'(x) > 0$ pour $x \in [-\frac{7}{3}, 3]$.

	$-\infty$		$-\frac{7}{3}$		3		$+\infty$
$f'(x)$		$-$	0	$+$	$\|\|$	$-$	
$f(x)$	1	$\searrow$	0.4375	$\nearrow +\infty, +\infty$		$\searrow$	1

$$f(-\tfrac{7}{3}) = \frac{(-\frac{7}{3})^2+7}{(-\frac{7}{3}-3)^2} = \frac{\frac{49}{9}+7}{(-\frac{16}{3})^2} = \frac{\frac{49+63}{9}}{} \times \frac{9}{16^2} = \frac{112}{16^2}$$

$= 0.4375$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2+7}{(x-3)^2} = \lim_{x\to\infty} \frac{x^2+7}{x^2-6x+9} = 1$$

$$\lim_{x\to 3} f(x) = \frac{3^2+7}{0} = +\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2+7}{(x-3)^2} = 1$$

DS : TD3 :

TD4 :

TD5 :

9.

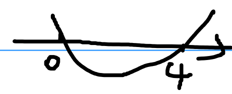
Ex6 : ② $f(x) = \dfrac{3x^2}{x-2}$. $D_f = \mathbb{R} \setminus \{2\}$.

$f'(x) = 3 \times \dfrac{2x \times (x-2) - x^2 \times 1}{(x-2)^2} = 3 \times \dfrac{2x^2 - 4x - x^2}{(x-2)^2}$

$= 3 \times \dfrac{x^2 - 4x}{(x-2)^2}$

$f'(x) = \dfrac{6x \times (x-2) - 3x^2 \times 1}{(x-2)^2} = \dfrac{6x^2 - 12x - 3x^2}{(x-2)^2} = \dfrac{3x^2 - 12x}{(x-2)^2}$.

$f'(x)$ a le signe du poly $x^2 - 4x = x(x-4)$, $(x-2)^2 > 0$ pour $x \in D_f$.



$f'(x) > 0$ pour $x \in]-\infty, 0] \cup [4, +\infty[$, donc $f(x)$ ↑

$f'(x) < 0$ pour $x \in [0, 4]$, donc $f(x)$ ↓

le tableau de variation.

x	$-\infty$		0		2		4		$+\infty$
$f'(x)$		$+$	0	$-$	‖ $+\infty$		0	$+$	
$f(x)$	$-\infty$	↗	0	↘ $-\infty$	‖ $+\infty$	↘	24	↗	$+\infty$

$f(0) = 0$ $\lim\limits_{x \to 2^+} f(x) = +\infty$ $f(4) = 24$ $\lim\limits_{x \to -\infty} f(x) = -\infty$

$\lim\limits_{x \to 2^-} f(x) = -\infty$ $\lim\limits_{x \to +\infty} f(x) = +\infty$